VARIATIONAL PROBLEMS OF OPTIMIZATION OF CONTROL PROCESSES FOR EQUATIONS WITH DISCONTINUOUS RIGHT-HAND SIDES

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In a variational formulation, the problems of optimization of control processes are discussed for systems described by differential equations with discontinuous right-hand sides. The necessary conditions of minimum of a functional are established and then applied to solution of the problems of optimization of the states of motion of oscillating conveyors.

1. Formulation of the problem. In an open region R of the $n + m$ dimensional space of the coordinates $x_1, \ldots, x_n$ and the control parameters $u_1, \ldots, u_m$, and in the interval of time $t_0 \leqslant t \leqslant T$, we have given the system of n ordinary differential equations of the first order

$$g_s^{\pm} = \dot{x}_s - f_s^{\pm}(x_1, \ldots, x_n, u_1, \ldots, u_m, t) = 0 \tag{1.1}$$

and the system of r finite relations

$$\psi_k^{\pm} = \psi_k^{\pm}(u_1, \ldots, u_m, t) = 0 \qquad (k = 1, \ldots, r < m) \tag{1.2}$$

The initial $x_s(t_0)$ and the final $x_s(T)$ values of the coordinates $x_s(t)$ are connected by the relations

$$\varphi_l = \varphi_l\,[x_1(t_0), \ldots, x_n(t_0), t_0, x_1(T), \ldots, x_n(T), T] = 0 \quad (l=1, \ldots, p \leqslant 2n+1) \tag{1.3}$$

The equation

$$\vartheta = \vartheta(x_1, \ldots, x_n, t) = 0 \tag{1.4}$$

determines the surface S which divides the region R (and the interval $t_0 \leqslant t \leqslant T$) into two parts: $R^-(t_0 \leqslant t < t')$ and $R^+(t' < t \leqslant T)$. In the part R^-, for which $\vartheta < 0$, Equations (1.1) and (1.2) are valid with the lower indices – assigned to the functions f_s and ψ_k, while in the part R^+, for which $\vartheta > 0$, the same equations hold with the upper indices + assigned to f_s and ψ_k.

The functions f_s^+ and f_s^- are continuous and have continuous derivatives of the order necessary for this discussion. They may be different and, when crossing the surface S, they may have discontinuities of the first kind. Similar assumptions are made for the functions ψ_k^+ and ψ_k^-.

Equations (1.1) are sometimes written in the form $\dot{x}_s = f_s^{\pm}$ and, therefore, we shall call them the equations with discontinuous right-hand sides.

We shall formulate the following optimization problem for Equations (1.1) and (1.2). From the functions $x_s(t)$, $(s = 1, \ldots, n)$, and $u_k(t)$, $(k = 1, \ldots, m)$, which satisfy Equations (1.1) and (1.2) with the indices + or – in the regions R^+ and R^-, respectively, and the quantities t_0, t_i' and T related by Equations (1.3) and (1.4), determine these which correspond to the minimum (or the maximum) of the functional

$$J = g\,[x_1(t_0), \ldots, x_n(t_0), t_0, x_1(T), \ldots, x_n(T), T] + \\ + \int_{t_0}^{T} f_0^{\pm}(x_1, \ldots, x_n, u_1, \ldots, u_m, t)\,dt \tag{1.5}$$

where the indices $\pm$ have the meaning explained above. Here and in the following considerations, it is assumed that the admissible arcs intersect the surface (1.4).

The times t_0, t_i', or T may be given in advance. Thus, the times of the beginning or of the end of the process, or the times of the discontinuities of the right-hand sides of the equations will be fixed in this problem.

In this formulation, the problem becomes the variational problem of Mayer and Bolza [1], complicated by the existence of the controls $u_k(t)$, whose derivatives do not enter into the equations of the problem, and in an essential way by the discontinuities of the right-hand sides of the equations of motion. The analysis of optimum setting of oscillating conveyors [2], vibratory pile drivers [3], etc., belong to this class of problems.

Another typical property of the optimization problems of control processes is the boundedness of the region of admissible variations of controls, which has not been discussed in this formulation. Such problems can be easily reduced to our formulation by the methods described in

[4.5]. Those methods make use of the equations of the type (1.2).

The existence of limitations imposed on the controls leads to the necessity of considering discontinuous solutions of the optimization problem. Therefore, the functions corresponding to an extremum of the functional J will be sought as continuous functions $x_s(t)$ with piece-wise continuous derivatives $\dot{x}_s(t)$ and piece-wise continuous controls $u_k(t)$.

In the following, the problem of minimum of the functional J will be considered. The case of maximum may be obtained by changing the sign of the functional.

In this paper, special attention is paid to the modifications introduced to the optimization problems by the discontinuities of right-hand sides of the equations of motion. The existence of these discontinuities requires special investigation in order to clarify the applicability of the known theorems and methods of the calculus of variations.

This investigation is accomplished by the methods which are similar to those in the book by Bliss [1]. Unfortunately, they prove to be too complicated for their full presentation in this paper. Therefore, we have been compelled to limit ourselves to the formulation of the theorems and rules used here, and to a short explanation concerning their proofs.

Only two necessary conditions of minimum of the functional, which are widely used in the optimization problems of control processes, are presented. They are the necessary condition of extremum and the necessary condition of Weierstrass of a strong minimum. Clebsch's condition of a weak minimum, which can be easily obtained from the Weierstrass condition [5], and Jacobi's necessary condition, requiring a complicated proof, are not given in this paper as they are seldom used in the optimization problems.

2. The condition of extremum of the functional J. It is shown in the Appendix that one of the necessary conditions of minimum of the functional J, i.e. the condition of extremum, is the zero value of the first variation ΔI of the functional I, which is constructed according to the formula

$$I = \varphi + \sum_{i=1}^{q} \nu_i \vartheta\,[x_1(t_i'), \ldots, x_n(t_i'), t_i'] + \int_{t_0}^{T} L\,dt \tag{2.1}$$

where

$$\varphi = g + \sum_{l=1}^{p} \rho_l \varphi_l \tag{2.2}$$

$$L = f_0^{\pm} + \sum_{s=1}^{n} \lambda_s g_s^{\pm} - \sum_{k=1}^{r} \mu_k \psi_k^{\pm} = \sum_{s=1}^{n} \lambda_s \dot{x}_s - H \tag{2.3}$$

$$H = H_\lambda + H_\mu = \sum_{s=0}^{n} \lambda_s^{\pm} f_s^{\pm} + f_s^{\pm} \sum_{k=1}^{r} \mu_k^{\pm} \psi_k^{\pm} \; (\lambda_0 = -1) \tag{2.4}$$

Here, ρ_l, ν_i, $\lambda_s(t)$ and $\mu_k(t)$ are the undetermined multipliers of Lagrange which should be calculated. Here and in the following, the indices $\pm$ are omitted in the cases which exclude any misunderstanding. The symbols δ and Δ denote the "variations at the point" and the "variations of the point", respectively. The difference between these two concepts has been explained in [4]. The summation in the second term on the right-hand side of the Expression (2.1) should be carried over all $i = 1, \ldots, q$, where q denotes the number of instants of time $t = t'$ for which the right-hand sides of the equations of motion are discontinuous.

To avoid confusion, we shall denote by $t = t^*$ the instants of discontinuities of the controls $u_k(t)$, and by $t = t'$ the instants of discontinuities of the right-hand sides of the equations of motion. The number q includes the instants $t = t'$ of discontinuities of the right-hand sides of the equations of motion at continuous controls as well as the instants $t = t'^*$ of the discontinuities of the right-hand sides of the equations of motion and the discontinuities of the controls $u_k(t)$.

We shall consider first the instants of the discontinuities of the right-hand sides of the equations of motion. For simplicity we assume that in the interval $t_0 \leqslant t \leqslant T$ only one point $t = t'$ of discontinuities of the functions f_s and ψ_k exists. Furthermore, to be specific, we assume that in the sub-interval $t_0 \leqslant t < t'$ the representing point belongs to the subregion R^- of the region R. Using in addition the Expression (2.3), we can represent the functional (2.1) in the form

$$I = \varphi + \nu\vartheta + \int_{t_0}^{t'} \Big(\sum_{s=1}^{n} \lambda_s^- \dot{x}_s^- - H^- \Big) dt + \int_{t'}^{T} \Big(\sum_{s=1}^{n} \lambda_s^+ \dot{x}_s^+ - H^+ \Big) dt \tag{2.5}$$

Constructing its first variation ΔI, we have

$$\begin{aligned}\Delta I = {} & \Delta\varphi + \nu\Delta\vartheta + (f_0^- - f_0^+)_{t'}\,\delta t' - (f_0)_{t_0}\delta t_0 + (f_0)_T\,\delta T + \\ & + \int_{t_0}^{t'} \Big\{ \sum_{s=1}^{n} \Big(\lambda_s^- \delta \dot{x}_s^- - \frac{\partial H^-}{\partial x_s^-} \delta x_s^- \Big) - \sum_{k=1}^{m} \frac{\partial H^-}{\partial u_k^-} \delta u_k^- \Big\} dt + \\ & + \int_{t'}^{T} \Big\{ \sum_{s=1}^{n} \Big(\lambda_s^+ \delta \dot{x}_s^+ - \frac{\partial H^+}{\partial x_s^+} \delta x_s^+ \Big) - \sum_{k=1}^{m} \frac{\partial H^+}{\partial u_k^+} \delta u_k^+ \Big\} dt\end{aligned} \tag{2.6}$$

In this, the following notations are introduced

$$\triangle \varphi = \frac{\partial \varphi}{\partial t_0} \delta t_0 + \frac{\partial \varphi}{\partial T} \delta T + \sum_{s=1}^{n} \left(\frac{\partial \varphi}{\partial x_s(t_0)} \triangle x_s(t_0) + \frac{\partial \varphi}{\partial x_s(T)} \triangle x_s(T) \right) \quad (2.7)$$

$$\triangle \vartheta = \frac{\partial \vartheta}{\partial t'} \delta t' + \sum_{s=1}^{n} \frac{\partial \vartheta}{\partial x_s(t')} \triangle x_s(t') \quad (2.8)$$

and the equations are used

$$\dot{x}_s^{\pm} = \frac{\partial H^{\pm}}{\partial \lambda_s^{\pm}} \quad (s = 1, \ldots, n), \qquad \frac{\partial H^{\pm}}{\partial \mu_k^{\pm}} = 0 \quad (k = 1, \ldots, r) \quad (2.9)$$

which are equivalent to Equations (1.1) and (1.2) and the relations (1.3) and (1.4). On the right-hand side of (2.6) the symbol $(f_0^-)t'$, for example, denotes that the value of the function f_0^- should be calculated for the time $t = t'$.

Integrating by parts the first sums of the integrands in the Expression (2.6), we obtain

$$\begin{aligned} \int_{t_0}^{t'} \sum_{s=1}^{n} \lambda_s^- \delta\dot{x}_s^- dt &= \sum_{s=1}^{n} \left\{ \lambda_s^-(t') \delta x_s^-(t') - \lambda_s^-(t_0) \delta x_s^-(t_0) - \int_{t_0}^{t'} \dot{\lambda}_s^- \delta x_s^- dt \right\} \\ \int_{t'}^{T} \sum_{s=1}^{n} \lambda_s^+ \delta\dot{x}_s^+ dt &= \sum_{s=1}^{n} \left\{ \lambda_s^+(T) \delta x_s^+(T) - \lambda_s^+(t') \delta x_s^+(t') - \int_{t'}^{T} \dot{\lambda}_s^+ \delta x_s^+ dt \right\} \end{aligned} \quad (2.10)$$

Using these together with the Formula (2.7), (2.8), and

$$\triangle x_s(t) = \delta x_s(t) + \dot{x}_s(t) \delta t \quad (2.11)$$

where t assumes the values t', t_0, T, we can transform the first variation (2.6) of the functional I to the following form

$$\begin{aligned} \triangle I = &\left[\frac{\partial \varphi}{\partial t_0} + \sum_{s=1}^{n} \lambda_s(t_0) \dot{x}_s(t_0) - (f_0)_{t_0} \right] \delta t_0 + \left[\frac{\partial \varphi}{\partial T} - \sum_{s=1}^{n} \lambda_s(T) \dot{x}_s(T) + (f_0)_T \right] \delta T + \\ &+ \left\{ \nu \frac{\partial \vartheta}{\partial t'} + (f_0^-)_{t'} - (f_0^+)_{t'} - \sum_{s=1}^{n} [\lambda_s^-(t') \dot{x}_s^-(t') - \lambda_s^+(t') \dot{x}_s^+(t')] \right\} \delta t' + \\ &+ \sum_{s=1}^{n} \left[\frac{\partial \varphi}{\partial x_s(t_0)} - \lambda_s(t_0) \right] \triangle x_s(t_0) + \sum_{s=1}^{n} \left[\frac{\partial \varphi}{\partial T} + \lambda_s(T) \right] \triangle x_s(T) + \\ &+ \sum_{s=1}^{n} \left[\lambda_s^-(t') - \lambda_s^+(t') + \nu \frac{\partial \vartheta}{\partial x_s(t')} \right] \triangle x_s(t') - \end{aligned}$$

$$-\int_{t_0}^{t'}\left\{\sum_{s=1}^{n}\left(\dot{\lambda}_s^- + \frac{\partial H^-}{\partial x_s^-}\right)\delta x_s^- + \sum_{k=1}^{m}\frac{\partial H^-}{\partial u_k^-}\delta u_k^-\right\}dt -$$
$$-\int_{t'}^{T}\left\{\sum_{s=1}^{n}\left(\dot{\lambda}_s^+ + \frac{\partial H^+}{\partial x_s^+}\right)\delta x_s^+ + \sum_{k=1}^{m}\frac{\partial H^+}{\partial u_k^+}\delta u_k^+\right\}dt \qquad (2.12)$$

This quantity should be equal to zero. In order to satisfy this requirement, it is necessary to follow the following procedure.

We select the multipliers $\lambda_s(t)$ in such a way that they satisfy the differential equations

$$\dot{\lambda}_s^{\pm} + \frac{\partial H^{\pm}}{\partial x_s^{\pm}} = 0 \qquad (s = 1, \ldots, n) \qquad (2.13)$$

The coefficients of the $2(m - r)$ independent variations $\delta u_k^{\pm}$ should be equal to zero. The remaining $2r$ coefficients of the dependent variations $\delta u_k^{\pm}$ become zero through the selection of the $2r$ multipliers $\mu_k^{\pm}$.

We have thus

$$\frac{\partial H^{\pm}}{\partial u_k^{\pm}} = 0 \qquad (k = 1, \ldots, m) \qquad (2.14)$$

The coefficients of the $2n + 2 - p$ independent variations of the set δt_0, $\Delta x_s(t_0)$, δT, $\Delta x_s(T)$ should be equal to zero. Selecting the multipliers ρ_l in such a way that the coefficients of the remaining p variations become zeros, we obtain the equations

$$\frac{\partial \varphi}{\partial t_0} + \sum_{s=1}^{n}\lambda_s(t_0)\,\dot{x}_s(t_0) - (f_0)_{t_0} = 0, \quad \frac{\partial \varphi}{\partial T} - \sum_{s=1}^{n}\lambda_s(T)\,\dot{x}_s(T) + (f_0)_T = 0 \qquad (2.15)$$

$$\frac{\partial \varphi}{\partial x_s(t_0)} - \lambda_s(t_0) = 0 \quad (s = 1, \ldots, n), \qquad \frac{\partial \varphi}{\partial x_s(T)} + \lambda_s(T) = 0 \quad (s = 1, \ldots, n) \qquad (2.16)$$

Finally, the set of variations $\delta t'$, $\Delta x_s(t')$ is related by one relation. Therefore, the coefficients of the n independent variations are equal to zero, and the multiplier ν can be selected such that the last coefficient of the dependent variation becomes equal to zero. In this way we find

$$\lambda_s^-(t') - \lambda_s^+(t') + \nu\frac{\partial \vartheta}{\partial x_s(t')} = 0 \qquad (s = 1, \ldots, n) \qquad (2.17)$$

$$\nu \frac{\partial \vartheta}{\partial t'} + (f_0^-)_{t'} - (f_0^+)_{t'} - \sum_{s=1}^{n} [\lambda_s^-(t')\, \dot{x}_s^-(t') - \lambda_s^+(t')\, \dot{x}_s^+(t')] = 0 \quad (2.18)$$

This system of equations replaces the usual conditions of Erdmann and Weierstrass.

The system of Equations (2.13) to (2.18) represents the condition of extremum of the functional J. In order to solve the optimization problem, this system should be supplemented by Equations (2.9), the relations (1.3) and (1.4), and by the conditions of continuity of the coordinates

$$x_s^-(t') = x_s^+(t') \qquad (s = 1, \ldots, n) \qquad (2.19)$$

In this way, with the above assumptions concerning the number of the points of discontinuity of the right-hand sides, the $4n + 2m + 2r$ functions $x_s^{\pm}(t)$, $\lambda_s^{\pm}(t)$, $u_k^{\pm}(t)$, and $\mu_k^{\pm}(t)$ are determined by the $4n + 2m + 2r$ Equations (2.9), (2.13) and (2.14). The integration of the $4n$ first-order equations (Equations (2.13) and the first group of Equations (2.9)) introduces $4n$ arbitrary constants. In order to determine these constants, together with the $p + 1$ multipliers ρ_l and ν, and the quantities t_0, t', and T, we use the $4n + p + 4$ conditions (2.15) to (2.19), (1.3) and (1.4).

Substituting the values of $\lambda_s(t_0)$ and $\lambda_s(T)$ from Equations (2.16) into Equations (2.15), we can write them in the form

$$\frac{d\varphi}{dt_0} = \frac{\partial \varphi}{\partial t_0} + \sum_{s=1}^{n} \frac{\partial \varphi}{\partial x_s(t_0)}\, \dot{x}_s(t_0) = (f_0)_{t_0} \qquad (2.20)$$

$$\frac{d\varphi}{dT} = \frac{\partial \varphi}{\partial T} + \sum_{s=1}^{n} \frac{\partial \varphi}{\partial x_s(T)}\, \dot{x}_s(T) = -(f_0)_T \qquad (2.21)$$

Substituting now the derivatives x from Equations (1.1) and using the notations (2.4), with the identity $H_\mu \equiv 0$, we obtain the relations

$$\partial\varphi/\partial t_0 = -(H)_{t_0}, \qquad \partial\varphi/\partial T = (H)_T \qquad (2.22)$$

Analogous transformations may be applied to the condition (2.18) resulting in the relation

$$\nu \frac{\partial \vartheta}{\partial t'} + (H^+)_{t'} - (H^-)_{t'} = 0 \qquad (2.23)$$

We shall consider now the discontinuities of the controls $u_k(t)$. We assume again that only one point $t = t^*$ of discontinuity of the controls exists in the interval $t_0 \leqslant t \leqslant T$. The right-hand sides of the equations of motion will be at first assumed to be continuous. Thus, we may use all the results obtained in [4,5] for the optimization problems of control processes for the equations of motion with continuous right-hand sides. Comparing them with the relations derived above, we see that Equations (2.9), (2.13) and (2.14), and the conditions (2.15), (2.16), (2.19) and (1.3) remain valid. Equation (1.4) should be neglected, and the Erdmann-Weierstrass conditions assume the following form

$$\lambda_s^-(t^*) - \lambda_s^+(t^*) = 0 \qquad (s = 1, \ldots, n), \qquad (H^-)_{t^*} - (H^+)_{t^*} = 0 \quad (2.24)$$

Here, the indices - and + denote that the functions belong to the intervals $t_0 \leqslant t \leqslant t^*$ and $t^* \leqslant t \leqslant T$, respectively.

If we assume that in the interval $t_0 \leqslant t \leqslant T$ there exists only one point $t = t'^*$ of discontinuity of the right-hand sides of the equations of motion and the controls $u_k(t)$, then the corresponding calculations result in the equations and conditions which coincide with those derived for the case of the point $t = t'$ of discontinuity of the equations of motion only.

The calculation of the number of functions and constants which should be determined, and of the number of equations and conditions obtained from the condition of extremum is performed in the same way as it was done This calculation shows that, in the last two cases, the number of equations and conditions is exactly sufficient for constructing the solution satisfying the condition of extremum of the functional J.

More complicated problems, with the curve corresponding to a minimum of the functional J having several corner points in the interval $t_0 \leqslant t \leqslant T$, will not be considered here. Such problems do not introduce any changes in the relations given above, but they strongly complicate the process of derivation.

Examining Equations (2.24), (2.17) and (2.18), we arrive at the conclusion that the points $t = t'$ and $t = t'^*$, which correspond to the discontinuities of the right-hand sides of the equation of motion, differ essentially from the point $t = t^*$ of the discontinuity of the controls $u_k(t)$. In fact, for $t = t^*$ the Lagrangian multipliers $\lambda_s(t)$ and the function H are continuous, while for $t = t'$ and $t = t'^*$ these functions may have discontinuities [7].

It is necessary to make one more important remark. If the equation of the surface S does not contain time t explicitly, then the function H, according to (2.23), will be continuous, even if λ_s may prove to be discontinuous.

If, in addition, the functions f_s and ψ_k do not depend on time, the system of equations derived above has the first integral

$$H = H_\lambda = h = \text{const} \tag{2.25}$$

and instead of the conditions (2.15) we have in this case

$$\partial\varphi / \partial t_0 = -h, \qquad \partial\varphi / \partial T = h \tag{2.26}$$

as implied by the relations (2.22).

3. The necessary condition of Weierstrass. Having determined a solution satisfying the condition of extremum, it is necessary to verify whether the functional J assumes its minimum value for this solution. Considering the discontinuities of the solution, we have to use the condition of Weierstrass for the absolute minimum of the functional J.

The formulation of the necessary condition of Weierstrass is given in the Appendix. It is constructed with the use of the Weierstrass function E which, in our problems, has the following form

$$\begin{aligned} E = L(x_1, \ldots, x_n, \dot{X}_1, \ldots, \dot{X}_n, U_1, \ldots, U_m, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_r, t) - \\ - L(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, u_1, \ldots, u_m, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_r, t) - \\ - \sum_{s=1}^{n} (\dot{X} - \dot{x}_s) \frac{\partial L}{\partial \dot{x}_s} \end{aligned} \tag{3.1}$$

where x_s and u_k are the coordinates and controls corresponding to the minimum of the functional J, and X_s and U_k are arbitrary admissible functions satisfying Equations (2.9). The function L may be discontinuous, but at the point of discontinuity $t = t'$ it has the left and the right limits.

The necessary condition of Weierstrass for the absolute minimum of the functional J is formulated in the form of the inequality

$$E \geqslant 0 \tag{3.2}$$

At the points of discontinuity of the right-hand sides of the equations of motion, this inequality should be satisfied by both limits of the function E at the discontinuities. Substituting L from the Expression (2.3) into (3.1) we obtain the following relation

$$\begin{aligned} E = H(x_1, \ldots, x_n, U_1, \ldots, U_m, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_r, t) - \\ - H(x_1, \ldots, x_n, u_1, \ldots, u_m, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_r, t) \end{aligned} \tag{3.3}$$

Considering that $H_\mu \equiv 0$, the condition (3.2) may be replaced by the inequality

$$H_\lambda(x_1, \ldots, x_n, U_1, \ldots, U_m, \lambda_1, \ldots, \lambda_n, t) \leqslant \\ \leqslant H_\lambda(x_1, \ldots, x_n, u_1, \ldots, u_m, \lambda_1, \ldots, \lambda_n, t) \tag{3.4}$$

which, at the points $t = t'$ and $t = t'^*$, holds for the left and right limits of the function H_λ.

This complication does not exist if the function ϑ in Equation (1.4), which determines the surface S of the discontinuities of the right-hand sides of the equations of motion, does not contain time t explicitly. Then, the function H_λ is continuous in the total interval $t_0 \leqslant t \leqslant T$. The book [7] contains a statement which indicates that the solution of the optimization problems for the equations of motion with discontinuous right-hand sides may be constructed by the methods following from the maximum principle of Pontriagin.

4. Example. A simple problem of the process of oscillating transport. A material particle B rests on a rough horizontal plane A (Fig. 1). The plane performs periodic horizontal vibrations with a given period T_0. For certain motions of this type, the particle B moves on the plane A. It is necessary to find the periodic motion of the plane A for which the average velocity of the particle B during one period T_0 reaches its maximum value, i.e. it is larger than for any other motion of the plane with the same period T_0.

We introduce the coordinate axes y and z connected with the plane A, as shown on Fig. 1, and we denote by $\xi(t)$ the displacement of the plane. The equation of motion of the particle B has then the form

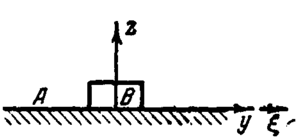

Fig. 1.

$$\ddot{y} = -\ddot{\xi} - fg\,\mathrm{sign}\,\dot{y} \qquad \text{or} \qquad g_1^{\pm} = \dot{x} + u \pm fg = 0 \tag{4.1}$$

where

$$x = \dot{y}, \qquad \ddot{\xi} = u \tag{4.2}$$

The equation of the "surface" of discontinuity of the right-hand side is

$$\vartheta = x = 0 \tag{4.3}$$

We shall assume that $u(t)$ is bounded in its absolute value

$$|u(t)| \leqslant U^* \tag{4.4}$$

If the absolute value of the acceleration $u(t)$ does not exceed the value fg, i.e.

$$|u(t)| \leqslant fg \tag{4.5}$$

then the point B moves on the plane with stops of finite duration. For these stops, the equation holds

$$g_1^{\circ} = x = 0 \tag{4.6}$$

In order to separate these two possible cases of motion we shall use the following convention. If the Equation (4.1) and the inequalities (4.4) are satisfied, we say that the particle B is in the zone of motion; if the Equation (4.6) and the inequality (4.5) are satisfied, the particle is in the zone of rest. We shall impose the essential requirement $U^* > fg$.

We investigate a periodic motion of the point B. Thus, the initial and the final velocities are related by the equation

$$\varphi_1 = x(T) - x(t_0) = 0 \tag{4.7}$$

In addition we have the relations

$$\varphi_2 = t_0 = 0, \quad \varphi_3 = T - T_0 = 0 \tag{4.8}$$

and

$$\varphi_4 = \int_0^{T_0} u(t)dt = 0 \tag{4.9}$$

which express the definiteness of the beginning and end of a period and the periodicity of the function $u(t)$.

Passing to the open regions of variability of "controls" in both zones, we write the relations

$$\psi^{\pm} = u - \chi(v) = 0 \tag{4.10}$$
$$\psi^{\circ} = u - \chi^{\circ}(v) = 0 \tag{4.11}$$

The diagrams of the functions $\chi(v)$ and $\chi^{\circ}(v)$ are shown in Fig. 2.

The optimization problem of the process of oscillating transport may be now formulated in the following form.

It is necessary to find such functions $x(t)$ and $u(t)$ satisfying Equations (4.1) and (4.10) in the zone of motion, Equations (4.6) and (4.11) in the zone of rest, and the conditions (4.3), (4.7) to (4.9), for which the functional

$$J = \int_0^{T_0} x(t)\, dt \tag{4.12}$$

has its maximum value.

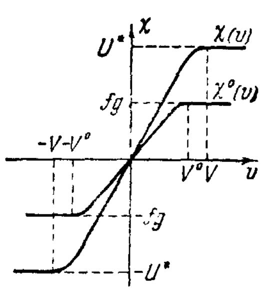

Fig. 2.

Here, instead of the average velocity of motion of the particle during one period $V = J/T_0$, we consider the displacement during the same period T_0. The problem is formulated for a maximum and, therefore, it is necessary to change the sign of inequality in the Weierstrass condition.

The optimization problem in the formulation described above is more complicated than those presented in previous sections. In addition to the discontinuities of the right-hand side of the equation of motion, in this case we have also the transition from the zone of motion to the zone of rest, i.e. from the differential equation (4.1) to the algebraic relation (4.6). The problems of this type can be solved by the methods described above. The presentation of the corresponding generalizations would cause a considerable expansion of this paper. Therefore, it will not be given here, and only the fundamental results of that analysis, written for the simple example being considered, will be used.

We construct the functions H and φ, which are necessary for the solution of the problem. In the zone of motion we have [6]

$$H = -x(t) - \lambda(t)[u \pm fg] + \rho u + \mu(t)[u - \chi(v)] \tag{4.13}$$

while in the zone of rest the function H is given by the formula

$$H^\circ = \rho u + \mu^\circ(t)[u - \chi^\circ(v)] \tag{4.14}$$

The function H is continuous in the whole interval $t_0 \leqslant t \leqslant T$. Furthermore, we have

$$\varphi = \rho_1[x(T) - x(t_0)] + \rho_2 t_0 + \rho_3(T - T_0) \tag{4.15}$$

According to the relations derived in Section 2, we construct the following equations

$$\dot{\lambda}^{\pm} = 1, \ -\lambda(t) + \rho + \mu(t) = 0, \ -\mu(t)\chi'(v) = 0 \tag{4.16}$$

which are valid in the zone of motion, and the equations

$$\rho + \mu(t) = 0, \ \ -\mu(t)\chi^{\circ}(v) = 0 \tag{4.17}$$

which should be satisfied in the zone of rest. In addition to these equations, the multipliers $\lambda(t)$, ρ_1, ρ_2, and ρ_3 are related by the conditions

$$\lambda(0) = \lambda(T_0) = -\rho_1, \ \ -\rho_2 = \rho_3 = h \tag{4.18}$$

where h is the constant from the equation $H = h$, which determines the first integral of the equations of the problem. In the zone of rest, $\lambda(t)$ remains undetermined.

The Weierstrass condition for both zones can be written in the form of the inequality

$$\mu(t)[U-u]\leqslant 0 \tag{4.19}$$

where $\mu(t)$ denotes the function determined by the second of Equations (4.16) or the first of Equations (4.17).

The first two Equations (4.16) show that in the zone of motion $\mu(t)$ is a linear function of time and becomes zero only at a finite number of points of the interval $t_0\leqslant t\leqslant T$. Therefore, $\chi'(v)=0$ and consequently

$$u=+U^* \quad \text{or} \quad u=-U^* \tag{4.20}$$

except for the points $t=t^*$, where $\mu(t^*)=0$. Analogously, we find that, for $\rho\neq 0$, in the zone of rest the following equations are valid

$$u=+fg, \quad \text{or} \quad u=-fg \tag{4.21}$$

These results simplify considerably the solution of the problem. Nevertheless, even taking them into account we have to consider a large number of solutions which might give the optimum process. Their construction would be an interesting illustration of the methods of analysis given above; however, the majority of these solutions either do not satisfy the Weierstrass condition or do not satisfy the conditions of periodicity (4.7) or (4.9). We shall not describe here these solutions, but we shall only consider the solution which gives the optimum process of oscillating transportation.

The periodic optimum solution of this simple problem of oscillating conveyor corresponds to the periodic function $u(t)$ whose diagram is given in Fig. 3. For definiteness, the origin of time $t=0$ is assumed as coinciding with the instant of discontinuity of the function $u(t)$, which varies from the value $+fg$ to the value $-U^*$. This solution can be easily constructed by the use of Equations (4.1) and (4.6) and the conditions of periodicity (4.7) and (4.9). It has the following form

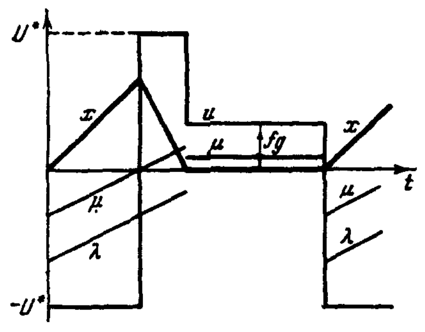

Fig. 3.

$$x(t)=(U^*-fg)t \qquad (0\leqslant t\leqslant t_1)$$

$$x(t)=(U^*+fg)t+2U^*t_1 \qquad (t_1\leqslant t\leqslant t_2)$$

$$x(t)=0 \qquad (t_2\leqslant t\leqslant T) \tag{4.22}$$

where

$$t_1 = \frac{U^* + fg}{U^*}\frac{T_0}{4}, \qquad t_2 = \frac{T_0}{2} \tag{4.23}$$

We shall construct now the Lagrangean multipliers $\lambda(t)$, $\mu(t)$ and ρ. For this purpose we construct the expression

$$\lambda(t) = -\rho_1 + t = \lambda^+(0) + t \tag{4.24}$$

satisfying the first of Equations (4.16) and the first of conditions (4.18). The second of Equations (4.18) results in $\lambda(T_0) = \rho_1$, and determines the discontinuity of the multiplier $\lambda(t)$:

$$\lambda^-(t_2) = \lambda^+(0) + \frac{T}{2} \tag{4.25}$$

At the time $t = t_1$ the equation $\mu(t_1) = 0$ holds, and it implies

$$\rho - \lambda^+(0) - \frac{U^* + fg}{U^*}\frac{T_0}{4} = 0 \tag{4.26}$$

These relations are valid in the zone of motion.

At the transition from the zone of motion to the zone of rest, the continuity condition of the function H gives

$$\lambda^-(t_2)[U^* + fg] = \rho[U^* - fg] \tag{4.27}$$

Equations (4.25) to (4.27) determine the quantities ρ, $\lambda^+(0)$ and $\lambda^-(t_2)$. Their solution yields the following values

$$\begin{gathered} \rho = -\frac{U^{*2} - (fg)^2}{fgU^*}\frac{T_0}{8} \\ \lambda^+(0) = -\frac{(U^* + fg)^2}{fgU^*}\frac{T_0}{8}, \qquad \lambda^-(t_2) = -\frac{(U^* - fg)^2}{fgU^*}\frac{T_0}{8} \end{gathered} \tag{4.28}$$

It is now easy to determine the multiplier $\mu(t)$ from Equations (4.16), (4.17) and (4.24) in the form

$$\begin{gathered} \mu(t) = t - \frac{U^* + fg}{U^*}\frac{T_0}{4} \qquad (0 \leqslant t \leqslant t_2) \\ \mu(t) = -\rho = \frac{U^{*2} - (fg)^2}{fgU^*}\frac{T_0}{8} \qquad (t_2 \leqslant t \leqslant T) \end{gathered} \tag{4.29}$$

The diagrams of the functions $\lambda(t)$ and $\mu(t)$ are shown in Fig. 3.

In the subinterval $0 \leqslant t \leqslant t_1$, we have $\mu(t) \leqslant 0$ and $u(t) = -U^*$. Therefore, the Weierstrass condition is satisfied for arbitrary admissible $u(t)$ satisfying the inequality $|u| \leqslant U^*$. In the next subinterval $t_1 \leqslant t \leqslant t_2$, we have $\mu(t) \geqslant 0$ and $u = +U^*$, and the Weierstrass condition is again satisfied for all admissible controls. Finally, in the last subinterval $t_2 \leqslant t \leqslant T$, it is $\mu(t) > 0$ and $u = +fg$. Thus, the Weierstrass condition is satisfied in the total interval $t_0 \leqslant t \leqslant T$.

A superficial consideration of the optimization problem of the process of oscillating transport may suggest that the optimum state of motion of a particle on the plane exists for the periodic displacement of the plane $\xi(t)$ having the acceleration diagram shown in Fig. 4. Therefore, concluding this paper we shall give the expressions for the displacement of the particle B during the period T_0 taking into account the optimum state of motion of the plane A as determined by previous analysis and the state of motion mentioned just now. These expressions have the following form

$$J_{\text{opt}} = \frac{U^{*2} - (fg)^2}{U^*} \frac{T_0^2}{16} \qquad (4.30)$$

$$J = \frac{U^* - fg}{U^* + fg} fg \frac{T_0^2}{4} \qquad (4.31)$$

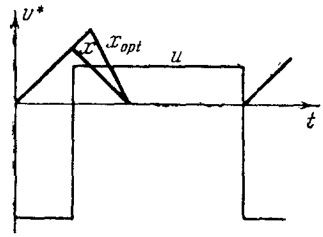

Fig. 4.

The first of them corresponds to the optimum state. For comparison, we calculate the difference between these two displacements

$$J_{\text{opt}} - J = \frac{T_0^2}{16} \frac{(U^* - fg)^3}{(U^* + fg)\, U^*} > 0 \quad \text{for } U^* > fg$$

The results are shown in Fig. 4, which contains the diagrams of velocities for both states of motion.

The author expresses his deep gratitude to A.I. Lur'e for his attention and interest to this paper, and to G.Iu. Dzhanelidze and I.I. Blekhman for their discussions of the results.

APPENDIX

In the $n + m$ dimensional space of the coordinates $x_1, \ldots, x_n$ and the controls $u_1, \ldots, u_m$, we shall consider a normal arc C satisfying Equations (1.1), (1.2) and the conditions (1.3), (1.4), and corresponding to a minimum of the functional J. On this arc, the controls $u(t)$ or the right-hand sides of the equations of motion may have a finite number of points of discontinuity. Such points will be called the corner points of the arc C.

We assume that, on the arc C, the matrix

$$\frac{\partial \psi}{\partial u} = \left\| \frac{\partial \psi_k}{\partial u_\beta} \right\|$$

whose k, β element is the derivative $\partial\psi_k/\partial u_\beta$ is of the rank r, equal to the number of Equations (1.2). Thus, using the arguments and calculations differing from those in the book by Bliss [1] only by the necessity of construction of the functions $u_k(b_1, \ldots, b_p, t)$ satisfying Equations (1.2), we can prove the following lemma.

If an admissible arc C satisfies Equations (1.1) and (1.2), and if p admissible sets of the constants and functions

$$\tau_{0\alpha}, \tau_{i'\alpha}, \tau_{T\alpha}, \xi_{s\alpha}(t), \zeta \quad (t) \tag{A.1}$$

are related by the variational equations on C

$$\dot{\xi}_{s\alpha} - \sum_{\gamma=1}^{n} \frac{\partial f_s}{\partial x_\gamma} \xi_{\gamma\alpha} - \sum_{\beta=1}^{m} \frac{\partial f_s}{\partial u_\beta} \zeta_{\beta\alpha} = 0, \qquad \sum_{\beta=1}^{m} \frac{\partial \psi_k}{\partial u_\beta} \zeta_{\beta\alpha} = 0 \tag{A.2}$$

where the derivatives $\partial f_s/\partial x_\gamma$, $\partial f_s/\partial u_\beta$, $\partial \psi_k/\partial u_\beta$ are taken on C, then there exists the p parametric family

$$\begin{array}{ll} x_s(b_1, \ldots, b_p, t) & (s = 1, \ldots, n), \\ u_k(b_1, \ldots, b_p, t) & (k = 1, \ldots, m), \end{array} \qquad t_0(b_1, \ldots, b_p) \leqslant t \leqslant T(b_1, \ldots, b_p) \tag{A.3}$$

containing C for $b_1 = \ldots = b_p = 0$ and consisting of the curves satisfying Equations (1.1) and (1.2). This family has the property that for any $\alpha = 1, \ldots, p$ the quantities (A.1) are equal to its variations with respect to b_α on C:

$$\tau_{0\alpha} = \left(\frac{\partial t_0}{\partial b_\alpha}\right)_0, \quad \tau_{i'\alpha} = \left(\frac{\partial t'}{\partial b_\alpha}\right)_0, \quad \tau_{T\alpha} = \left(\frac{\partial T}{\partial b_\alpha}\right)_0$$

$$\xi_{s\alpha} = \left(\frac{\partial x_s}{\partial b_\alpha}\right)_0, \quad \zeta_{k\alpha} = \left(\frac{\partial u_k}{\partial b_\alpha}\right)_0 \tag{A.4}$$

where the index 0 denotes that the derivatives are calculated for $b_1 = \ldots = b_p = 0$.

Let us assume that we have constructed a $p + q + 1$ parametric family of the curves (A.3). Substituting the functions (A.3) into the functional J we have

$$J = J\ (b_1, \ldots, b_{p+q+1}) \tag{A.5}$$

The total differential of this function is

$$dJ = f_0 \delta t \,|_{t_0}^{T} + \sum_{i=1}^{q} [f_0^-(t_i') - f_0^+(t_i')]\, \delta t_i' + dg +$$

$$+ \int_{t_0}^{T} \left[\sum_{s=1}^{n} \left(\frac{\partial f_0}{\partial x_s} \delta x_s + \frac{\partial f_0}{\partial \dot{x}_s} \delta \dot{x}_s\right) + \sum_{k=1}^{m} \frac{\partial f_0}{\partial u_k} \delta u_k\right] dt \tag{A.6}$$

where

$$\delta t_0 = \sum_{\alpha=1}^{p+q+1} \frac{\partial t_0}{\partial b_\alpha} db_\alpha, \qquad \delta t_i' = \sum_{\alpha=1}^{p+q+1} \frac{\partial t_i'}{\partial b_\alpha} db_\alpha, \qquad \delta T = \sum_{\alpha=1}^{p+q+1} \frac{\partial T}{\partial b_\alpha} db_\alpha$$
$$\delta x_s = \sum_{\alpha=1}^{p+q+1} \frac{\partial x_s}{\partial b_\alpha} db_\alpha, \qquad \delta u_k = \sum_{\alpha=1}^{p+q+1} \frac{\partial u_k}{\partial b_\alpha} db_\alpha \tag{A.7}$$

For the curve C this differential assumes the form

$$dJ = \sum_{\alpha=1}^{p+q+1} J_1(\xi_\alpha, \zeta_\alpha, \tau_\alpha)\, db_\alpha \tag{A.8}$$

where

$$J_1(\xi_\alpha, \zeta_\alpha, \tau_\alpha) = J_{1\alpha} = f_0 \tau_\alpha \big|_{t_0}^{T} + \sum_{i=1}^{q} (f_0^- - f_0^+)_{t_i'} \tau_{t_i'\alpha}' + G_{1\alpha} +$$
$$+ \int_{t_0}^{T} \left[\sum_{s=1}^{n} \left(\frac{\partial f_0}{\partial x_s} \xi_{s\alpha} + \frac{\partial f_0}{\partial \dot{x}_s} \dot{\xi}_{s\alpha} \right) + \sum_{k=1}^{m} \frac{\partial f_0}{\partial u_k} \zeta_{k\alpha} \right] dt \tag{A.9}$$

and $G_{1\alpha}$ is the linear form

$$G_{1\alpha} = G_1[\xi_{1\alpha}(t_0), \ldots, \xi_{n\alpha}(t_0), \tau_{0\alpha}, \xi_{1\alpha}(T), \ldots, \xi_{n\alpha}(T), \tau_{T\alpha}] =$$
$$= \frac{dg}{dt_0} \tau_{0\alpha} + \frac{dg}{dT} \tau_{T\alpha} + \sum_{s=1}^{n} \left[\frac{\partial g}{\partial x_s(t_0)} \xi_{s\alpha}(t_0) + \frac{\partial g}{\partial x_s(T)} \xi_{s\alpha}(T) \right] \tag{A.10}$$

The function $J_{1\alpha}$ is called [1] the first variation of the functional J with respect to b_α.

We introduce now into discussion the sum

$$L(x, \dot{x}, u, \lambda, \mu, t) = f_0 + \sum_{s=1}^{n} \lambda_s g_s - \sum_{k=1}^{r} \mu_k \psi_k \tag{A.11}$$

where $\lambda_s(t)$ and $\mu_k(t)$ are the Lagrange multipliers which should be determined. We also note that, if $\xi_{s\alpha}(t)$ and $\zeta_{k\alpha}(t)$ satisfy the variational equations (A.2), we can write

$$J_{1\alpha} = f_0 \tau_\alpha \big|_{t_0}^{T} + \sum_{i=1}^{q} (f_0^- - f_0^+)_{t_i'} \tau_{t_i'\alpha} + G_{1\alpha} +$$
$$+ \int_{t_0}^{T} \left[\sum_{s=1}^{n} \left(\frac{\partial L}{\partial x_s} \xi_{s\alpha} + \frac{\partial L}{\partial \dot{x}_s} \dot{\xi}_{s\alpha} \right) + \sum_{k=1}^{m} \frac{\partial L}{\partial u_k} \zeta_{k\alpha} \right] dt \tag{A.12}$$

Substituting the functions (A.3) into the left-hand sides of Equations (1.3) and (1.4), we obtain the functions $\varphi_l(b_1, \ldots, b_{p+q+1})$ and

$\vartheta_i(b_1, \ldots, b_{p+q+1})$ such that the equations

$$\begin{gathered} J(b_1, \ldots, b_{p+q+1}) = J(0, \ldots, 0) + u, \qquad \varphi_l(b_1, \ldots, b_{p+q+1}) = 0 \quad (l = 1, \ldots, p) \\ \vartheta\,[x_1(t_i'), \ldots, x_n(t_i'),\ t_i'] = \vartheta_i(b_1, \ldots, b_{p+q+1}) = 0 \qquad (i = 1, \ldots, q) \end{gathered} \tag{A.13}$$

have the solution $b_1 = \ldots = b_{p+q+1} = u = 0$ corresponding to the arc C. The functional determinant

$$\begin{vmatrix} \partial J/\partial b_\alpha \\ \cdots\cdots \\ \partial\varphi_l/\partial b_\alpha \\ \cdots\cdots \\ \partial\vartheta_i/\partial b_\alpha \end{vmatrix} = \begin{vmatrix} J_1(\xi_\alpha, \zeta_\alpha, \tau_\alpha) \\ \cdots\cdots \\ \Phi_l(\xi_\alpha, \zeta_\alpha, \tau_\alpha) \\ \cdots\cdots \\ \theta_i(\xi_\alpha, \zeta_\alpha, \tau_\alpha) \end{vmatrix} \tag{A.14}$$

where Φ_l and θ_i denotes the variations of the terminal conditions (1.3) and of the Equation (1.4)

$$\begin{gathered} \Phi_{l\alpha} = \Phi_l(\xi_\alpha, \zeta_\alpha, \tau_\alpha) = \frac{d\varphi_l}{dt_0}\tau_{0\alpha} + \frac{d\varphi_l}{dT}\tau_{T\alpha} + \sum_{s=1}^{n}\left[\frac{\partial\varphi_l}{\partial x_s(t_0)}\xi_{s\alpha}(t_0) + \frac{\partial\varphi_l}{\partial x_s(T)}\xi_{s\alpha}(T)\right] \\ \theta_{i\alpha} = \theta_i(\xi_\alpha, \zeta_\alpha, \tau_\alpha) = \left(\frac{d\vartheta}{dt}\right)_{t_i'}\tau_{t_i'\alpha} + \sum_{s=1}^{n}\left(\frac{\partial\vartheta}{\partial x_s(t)}\right)_{t_i'}\xi_{s\alpha}(t_i') \end{gathered} \tag{A.15}$$

must be equal to zero on C for arbitrary variations. In the opposite case, Equations (A.13) have the solutions $b_\alpha = B_\alpha(u)$ which become equal to zero for $u = 0$. Consequently, $J(u) < J(0)$ for $u < 0$, and $J(0)$ is not a minimum.

Therefore, the rank of the matrix of the determinant (A.14) does not exceed $p + q$, and the system of linear equations

$$J_{1\alpha} + \sum_{l=1}^{p}\rho_l\Phi_{l\alpha} + \sum_{i=1}^{q}\nu_i\theta_{i\alpha} = 0 \quad (\alpha = 1, \ldots, p + q + 1) \tag{A.16}$$

has non-trivial solutions ρ_l and ν_i. With these ρ_l and ν_i, the equation

$$J_1(\xi, \zeta, \tau) + \sum_{l=1}^{p}\rho_l\Phi_l(\xi, \zeta, \tau) + \sum_{i=1}^{q}\nu_i\theta_i(\xi, \tau) = 0 \tag{A.17}$$

should be satisfied for arbitrary admissible variations τ_0, $\tau_{t_i'}$, τ_T, $\xi_s(t)$, $\zeta_k(t)$.

Substituting now $J_1(\xi, \zeta, \tau)$ from (A.12) into Equations (A.17) we obtain the relation

$$f_0\tau\,|_{t_0}^{T}+\sum_{i=1}^{q}(f_0^- - f_0^+)_{t_i'}\tau_{t_i'}+G_1+\sum_{l=1}^{p}\rho_l\Phi_l+\sum_{i=1}^{q}\nu_i\theta_i+$$
$$+\int_{t_0}^{T}\left[\sum_{s=1}^{n}\left(\frac{\partial L}{\partial x_s}\xi_s+\frac{\partial L}{\partial \dot{x}_s}\dot{\xi}_s\right)+\sum_{k=1}^{m}\frac{\partial L}{\partial u_k}\zeta_k\right]dt \qquad (A.18)$$

which in the Lagrangean notations assumes the form

$$\triangle\left[g+\sum_{l=1}^{p}\rho_l\varphi_l+\sum_{i=1}^{q}\nu_i\vartheta\,(x_1(t_i'),\ \ldots,\ x_n(t_i'),\ t_i')+\int_{t_0}^{T}L\,dt\right]=0 \qquad (A.19)$$

or

$$\triangle I=0 \qquad (A.20)$$

where

$$I\doteq\varphi+\sum_{i=1}^{q}\nu_i\vartheta\,[x_1(t_i'),\ldots,x_n(t_i'),t_i']+\int_{t_0}^{T}L\,dt \qquad (A.21)$$

$$\varphi=g+\sum_{l=1}^{p}\rho_l\varphi_l \qquad (A.22)$$

The condition (A.20) should be satisfied for an arbitrary admissible arc C which satisfies Equations (1.1) and (1.2) and the conditions (1.3) and (1.4) and which corresponds to the minimum of the functional J. This condition may be called the condition of extremum of the functional J. It has been used in Section 2, where its expanded form was given.

The arguments and calculations leading to the necessary condition of Weierstrass for a strong minimum of the functional J, in the case of the problems with continuous right-hand sides of the equations of motion, are described in full details in the Appendix of [5]. They will not be repeated here; only the special aspects introduced by the discontinuities of the right-hand sides will be indicated.

At an arbitrary point of the arc C, not coinciding with the corners of C, the inequality should be satisfied

$$E\geqslant 0 \qquad (A.\ 23)$$

where

$$E=L\,(x,\dot{X},U,\lambda,\mu,t)-L\,(x,\dot{x},u,\lambda,\mu,t)-\sum_{s=1}^{n}(\dot{X}_s-\dot{x}_s)\frac{\partial L}{\partial \dot{x}_s} \qquad (A.\ 24)$$

while x_s, u_k correspond to the arc C, and X_s, U_k are arbitrary admissible functions satisfying Equations (1.1) and (1.2). The function E may have discontinuities at the points $t=t'$, where the right-hand sides of the equations of motion are discontinuous. However, it has the left and right limits at these points. Therefore, at the points of discontinuities of the right-hand sides of the equations of motion, the inequality (A.23) should be satisfied for both, the left and the right limits of the function E.

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